III A

1 (Ratio-test) Let (x_n) be a positive seq ($x_n > 0\ \forall n$) such that $\lim_n \frac{x_{n+1}}{x_n} = x < 1$.
Show that $\lim_n x_n = 0$.
(Take r s.t. $x < r < 1$, and take $N \in \mathbb{N}$ s.t. $\frac{x_{n+1}}{x_n} < r\ \forall n \geq N$. Show that $x_n < r^{n-N} x_N$ $\forall n > N$ and ~~note~~ that $r^{n-N} \to 0$ as $n \to \infty$).

2.* (Root-test). Similar as Q1 but assumption is replaced by $\lim_{n\to\infty} x_n^{1/n} = x < 1$.

3.* Show that $n \ll n^2 \ll 2^n \ll 100^n \ll n! \leq n^n$ where e.g. $n! \ll n^n$ is to be read as $\lim_n \frac{n!}{n^n} = 0$

4. Show that, as $n \to \infty$,
   a) $n^{1/n} \to 1$ (Hint: write $n^{1/n} = 1 + \delta_n$)
   b) $n^{1/n^2} \to 1$ (Hint: Squeeze Th)

5.* Let (x_n) be a positive sequence and let
$$S_n = \sum_{i=1}^{n} x_i \quad \forall n \in \mathbb{N}$$
Suppose $\exists$ a constant M $(\in \mathbb{R})$ s.t. $S_n \leq M\ \forall n$.
Show that the "series" $\sum_{n=1}^{\infty} x_n$ converges in the sense that $\lim_n S_n$ exists in $\mathbb{R}$.

6.* Let (x_n), (y_n) be positive sequences such that
$$x_n \leq y_n \quad \forall n.$$
Suppose $\sum_{n=1}^{\infty} y_n$ converges. Show that $\sum_{n=1}^{\infty} x_n$ also converges (Comparison Test of Series Convergence)

7.* Show that absolutely convergent series is convergent: $\sum_{n=1}^{\infty} |x_n| < +\infty \implies \sum_{n=1}^{\infty} x_n$ conv. (Hint: Cauchy Criterion)